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# Towards the QCD String: $2+1$ dimensional Yang-Mills theory in the planar limit

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**Abstract**

We study the large $N$ (planar) limit of pure $SU(N)$ 2+1 dimensional Yang-Mills theory ($YM_{2+1}$) using a gauge-invariant matrix parameterization introduced by Karabali and Nair. This formulation crucially relies on the properties of local holomorphic gauge invariant collective fields in the Hamiltonian formulation of $YM_{2+1}$. We show that the spectrum in the planar limit of this theory can be explicitly determined in the $N=\infty$, low momentum (large 't Hooft coupling) limit, using the technology of the Eguchi-Kawai reduction and the existing knowledge concerning the one-matrix model. The dispersion relation describing the planar $YM_{2+1}$ spectrum reads as $\omega(\vec{k}) = \sqrt{\vec{k}^2 + m_n^2}$, where $n = 1, 2, ...$ and $m_n = nm_r$, where $m_r$ denotes the renormalized mass, the bare mass $m$ being determined by the planar 't Hooft coupling $g_{YM}^2 N$ via $m = \frac{g_{YM}^2 N}{2\pi}$. The planar, low momentum limit, also captures the expected short and long distance physics of $YM_{2+1}$ and gives an interesting new picture of confinement. The computation of the spectrum is possible due to a reduction of the $YM_{2+1}$ Hamiltonian for the large 't Hooft coupling to the *singlet* sector of an effective one matrix model. The crucial observation is that the correct vacuum (the large $N$ master field), consistent with the area law and the existence of a mass gap, is described by an effective quadratic matrix model, in the large $N$, large 't Hooft coupling limit.

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# 1 Introduction

The study of the large $N$ limit of Yang-Mills theory is one of the grand problems of theoretical physics. In recent years, a new viewpoint has emerged concerning the planar limit of gauge theories, mainly motivated by recent insightful advances in string theory based on the duality between string and gauge theories [1]. Nevertheless, a precise formulation (a prerequisite for a solution) of the elusive "QCD string" is still lacking.

In this letter we take a fresh look at this problem in the setting provided by $2+1$ dimensional Yang Mills theory ($YM_{2+1}$) - a highly non-trivial quantum field theory [2], [3]. This theory is expected on many grounds to share the essential features of its $3+1$ dimensional cousin, such as asymptotic freedom and confinement, yet is distinguished from its $3+1$ dimensional counterpart by the existence of a dimensionful coupling. We regard this study as a stepping stone towards the $3+1$ dimensional theory.

Interestingly enough, we are able to make a precise statement concerning the spectrum of this theory in the large $N$, reduced, low momentum limit. In this limit, we argue, the generic features of the Yang-Mills vacuum are fully captured by a quadratic large $N$ matrix model. We perform explicit computations in a well-defined framework utilizing a momentum expansion of the reduced, planar effective action given in terms of the local gauge invariant variables which correspond to the only propagating physical polarization. Our approach utilizes many recipes from the large $N$ cookbook (the large $N$ reduction, matrix model technology), and yet is seemingly not *directly* related to the recent advances in the understanding of certain planar gauge theories from a string theory (gauge theory/gravity duality) point of view. This of course does not mean that a possible *indirect* connection is non-existent. We note that our approach can be understood as a target space, Hamiltonian formulation of an effective string field theory describing the planar $2+1$ dimensional QCD string.

Our work is crucially based on beautiful results derived in the remarkable work of Karabali and Nair [3]. They have provided an explicit gauge invariant reformulation of $YM_{2+1}$ in terms of local holomorphic variables. On a more practical level, Karabali and Nair have been able to compute the string tension in their Hamiltonian approach which is in excellent agreement (up to 3%) with the existing lattice data [4]. That striking result as well as the computation presented in this letter clearly point out that the Karabali-Nair approach has some truly remarkable features which can lead to potentially dramatic results in the arena of $2+1$ dimensional gauge theories.

It is reasonable to believe that if a solution of $YM_{2+1}$ is to be found at all, it will be in the planar limit. Consequently, we consider here *the large $N$ limit* of $YM_{2+1}$ using the Karabali-Nair parameterization, and this is the distinguishing feature of our work. We consider the spectrum of the theory and are able to show that there exists a mass gap set by the 't Hooft coupling. This result extends but is certainly consistent with the results of [3].

The crucial element in the computation of the spectrum is a large $N$ reduction of the $YM_{2+1}$ Hamiltonian, in a well-defined low momentum limit (large 't Hooft coupling), written in terms of the Karabali-Nair variables, to the *singlet sector of an effective one Hermitian matrix model*. The singlet sector is selected by the presence of a local holomorphic invariance of the planar vacuum which arises in the Karabali-Nair formalism. This reduction procedure enables us to write a self-consistent gap equation for the planar sector of $YM_{2+1}$. What is most important is that our approach provides a well-defined momentum expansion of the full effective local gauge invariant collective field theory of $YM_{2+1}$ in the reduced, planar limit. In particular, the correct Yang-Mills vacuum (*i.e.*, the large $N$ master field), consistent with the area law and the existence of a mass gap, is captured by an effective quadratic matrix model, in the large $N$, large 't Hooft coupling limit. This effective theory of gauge invariant holomorphic loop variables can be in principle used for other calculations, such as the determination of various correlation functions.

The outline of this paper is as follows: In Section 2 we review the Karabali-Nair variables and then in Section 3 we investigate in detail the Karabali-Nair collective field theory Hamiltonian. The large $N$ limit of this Hamiltonian is studied in Section 4 and the planar spectrum of the same in Section 5. A few more technical details related to the analysis of the collective field Hamiltonian are collected in a separate

Appendix at the end of this letter.

## 2 The Karabali-Nair variables

The Karabali-Nair approach can be summarized as follows [3]: consider an $SU(N)$ $YM_{2+1}$ in the Hamiltonian gauge $A_0 = 0$. Write the gauge potentials as $A_i = -it^a A_i^a$, for $i = 1, 2$, where $t^a$ are the Hermitian $N \times N$ matrices in the $SU(N)$ Lie algebra $[t^a, t^b] = if^{abc}t^c$ with the normalization $2Tr(t^a t^b) = \delta^{ab}$. Define complex coordinates $z = x_1 - ix_2$ and $\bar{z} = x_1 + ix_2$, and furthermore $2A^a = A_1^a + iA_2^a$, $2\bar{A}^a = A_1^a - iA_2^a$.

The Karabali-Nair parameterization is

$$A = -\partial_z M M^{-1}, \quad \bar{A} = +(M^{-1})^\dagger \partial_{\bar{z}} M^\dagger \tag{1}$$

where $M$ is a general element of $SL(N, \mathbb{C})$. Note that a (time independent) gauge transformation $A \to gAg^{-1} - \partial g\ g^{-1}$, $\bar{A} \to g\bar{A}g^{-1} - \bar{\partial} g\ g^{-1}$, where $g \in SU(N)$ becomes simply $M \to gM$. The variables $M$ correspond to holomorphic loops; their most important, and perhaps unexpected, property is locality! The corresponding *local gauge invariant* variables are given in terms of "closed loops" $H \equiv M^\dagger M$. Note that the standard Wilson loop operator may be written

$$\Phi(C) = TrPexp\{-i \oint_C dz\ \partial_z H H^{-1}\} \tag{2}$$

and thus is closely related to the local $H$ variables.

Now one might wonder whether the parameterization (1) is well-defined. In fact, the definition of $M$ implies a *holomorphic invariance*

$$\begin{aligned} M(z,\bar{z}) &\to M(z,\bar{z})h^\dagger(\bar{z}) && (3) \\ M^\dagger(z,\bar{z}) &\to h(z)M^\dagger(z,\bar{z}) && (4) \end{aligned}$$

where $h(z)$ is an arbitrary unimodular complex matrix whose matrix elements are independent of $\bar{z}$. Under the holomorphic transformation, the gauge invariant variable $H$ transforms homogeneously

$$H(z,\bar{z}) \to h(z)H(z,\bar{z})h^\dagger(\bar{z}) \tag{5}$$

This is distinct from the original gauge transformation, since it acts as right multiplication rather than left and is holomorphic. One way to understand its appearance is that the parameterization (1) can be formally inverted in the form

$$M(x,\bar{x}) = \left(1 - \int d^2z\ G(x,z)A_z(z,\bar{z}) + \ldots\right)\bar{V}(\bar{x}) \tag{6}$$

where $G$ is the Green's function, $\partial_z G(z,x) = \delta^{(2)}(z-x)$ and $\bar{V}$ is an arbitrary matrix with only anti-holomorphic dependence. The theory written in terms of the gauge invariant $H$ fields will have its own local (holomorphic) invariance. The gauge fields, and the Wilson loop variables, know nothing about this extra invariance. We will deal with this, as in [3], by requiring that the wave functions (or equivalently, physical states) be holomorphically invariant. The insistence of the holomorphic invariance of the vacuum is of crucial importance for our main argument in what follows.

One of the most remarkable properties of this parameterization is that the Jacobian relating the measures on the space of connections $C$ and on the space of gauge invariant variables $H$ can be explicitly computed

$$d\mu[C] = \sigma d\mu[H] e^{2c_A S_{WZW}[H]} \tag{7}$$

where $c_A$ is the quadratic Casimir in the adjoint representation of $SU(N)$ ($c_A = N$) and

$$S_{WZW}(H) = -\frac{1}{2\pi}\int d^2z\ TrH^{-1}\partial H H^{-1}\bar{\partial}H + \frac{i}{12\pi}\int d^3x\ \epsilon^{\mu\nu\lambda} TrH^{-1}\partial_\mu H H^{-1}\partial_\nu H H^{-1}\partial_\lambda H \tag{8}$$

is the level $-c_A$ hermitian Wess-Zumino-Witten action, which is both gauge and holomorphic invariant. $\sigma$ is a constant determinant factor. Thus the inner product may be written as an overlap integral of gauge invariant wave functionals with non-trivial measure

$$\langle 1|2\rangle = \int d\mu[H] e^{2c_A S_{WZW}(H)} \Psi_1^* \Psi_2 \tag{9}$$

The standard $YM_{2+1}$ Hamiltonian

$$\int Tr\left(g_{YM}^2 {E_i}^2 + \frac{1}{g_{YM}^2} B^2\right) \tag{10}$$

can be also explicitly rewritten in terms of gauge invariant variables. The collective field form [5] of this Hamiltonian (which we will refer to as the Karabali-Nair Hamiltonian) can be easily appreciated from its explicit form in terms of the natural "current"-like gauge invariant variables[1] $J = \frac{c_A}{\pi} \partial_z H H^{-1}$,

$$\mathcal{H}_{KN}[J] = m\left(\int_x J^a(x) \frac{\delta}{\delta J^a(x)} + \int_{x,y} \Omega_{ab}(x,y) \frac{\delta}{\delta J^a(x)} \frac{\delta}{\delta J^b(y)}\right) + \frac{\pi}{m c_A} \int_x \bar\partial J^a \bar\partial J^a \tag{11}$$

where

$$m = \frac{g_{YM}^2 c_A}{2\pi}, \quad \Omega_{ab}(x,y) = \frac{c_A}{\pi^2} \frac{\delta_{ab}}{(x-y)^2} - \frac{i}{\pi} \frac{f_{abc} J^c(x)}{(x-y)}. \tag{12}$$

The derivation of this Hamiltonian involves carefully regulating certain divergent expressions in a gauge invariant manner. We note that the scale $m$ is essentially the 't Hooft coupling.

At this point we remind the reader about the difference between collective field theory and effective field theory. Collective field theory [5] is simply based on a choice of collective variables, appropriate to the physics in question. The technical difficulty usually lies in the explicit change of variables, which generically renders the collective field theory horribly non-local. (This is, for example, the case with the collective field theory of canonical Wilson loop variables.) The crucial requirement that the large $N$ collective field theory has to meet is the factorization of vacuum correlators. The factorization in turn implies, by the resolution of the identity, that the only state that controls the physics at large $N$ is the vacuum. Notice that because of factorization at large $N$, one essentially has to be concerned with the appropriate classical phase space of gauge invariant observables and their canonically conjugate partners and correspondingly the classical Hamiltonian. The expectation values (evaluated using appropriate semiclassical coherent states) of the quantum Hamiltonian, lead to the required classical Hamiltonian, which in turn is nothing else but the collective field Hamiltonian [5].

The truly amazing feature of the Karabali-Nair holomorphic loop variables is that they are local and that the corresponding Jacobian can be explicitly computed! This Jacobian, determined in terms of the Wess-Zumino-Witten action, enjoys certain analyticity properties which render it unique and independent of regularization ambiguities.

The passage to the collective field Hamiltonian may be thought of as a starting point: having performed the change of variables, we may then analyze the theory using effective field theory techniques. In particular, one expects in general that there will be renormalizations; because of the local nature of the variables, one may expect that a suitable perturbative analysis can be found which sensibly deals with such matters. Indeed, we will describe such a formalism here and explain how the dynamics of the mass gap and confinement arises.

Finally, note that the inner product can be put into a canonical form

$$\langle 1|2\rangle = \int d\mu[H] \Phi_1^* \Phi_2 \tag{13}$$

provided we perform a redefinition of wavefunctionals $\Phi = e^{c_A S_{WZW}(H)} \Psi$. In so doing, there will be a corresponding adjustment of the collective Hamiltonian, containing new terms. We will display this explicitly

[1]The $J$ variables transform as connections under the holomorphic transformation.

in the following sections, but note here that the most important effect is to add a term of the form[2] $m^2 Tr(\partial H \bar{\partial} H^{-1})$, which will later be understood to correspond to the appearance of the mass gap in the large $N$ limit. Of course, near $g_{YM} \to 0$, we would take the perturbative vacuum and this term is of no particular relevance — the description of the theory in terms of $A$ and $\bar{A}$ is adequate. However, this term has arisen from the Jacobian of the path integral measure and although the change of variables in the measure is essentially an operation on the *classical* configuration space, we claim that one is lead to address the physical *non-perturbative* vacuum. It is with respect to this vacuum that the mass gap appears.

One notes that this collective field formalism is true for any rank of the gauge group, and in particular agrees with the large $N$ 't Hooft counting. We will obtain additional insight into the dynamics of confinement however by examining the theory in the large $N$ limit. First, let us continue reviewing the results of Refs. [3].

### 2.1 The vacuum wave functional, area law and string tension

One of the major results of the Karabali-Nair collective field theory approach is the analytic deduction of the area law and an explicit computation of the string tension [3].

The computation of the string tension is achieved by an approximate formula for the vacuum wave functional $\Psi$. First, one notices that $\Psi = 1$ is annihilated by the kinetic term and is normalizable given the non-trivial measure (due to the normalization of the WZW path integral).

One may find a vacuum wave functional annihilated by the total collective Hamiltonian ($\mathcal{H}\Psi \equiv (T + V)\Psi = 0$) by expanding this equation in powers of (roughly) $B/m^2$. To leading order, one finds

$$\Psi \simeq \exp\left[-\frac{1}{2g_{YM}^2}\int B(x)\left(\frac{1}{m+\sqrt{m^2-\nabla^2}}\right)B(y)\right] \tag{14}$$

Note that this wavefunctional apparently interpolates between the low and high momenta regions. At high momenta, this wavefunction correctly has a form corresponding to free gluons $\Psi \sim e^{-\frac{1}{2g_{YM}^2}\int B^2/k}$, appropriate to the conformally invariant two-point function of gluons, $\langle AA\rangle \sim g_{YM}^2/|k|$. In the low momentum region, the momentum factor is cut-off, and $B^2/k \to B^2/m$. Although higher order corrections are expected to be non-local, this can make sense self-consistently, if the theory may be re-organized into an expansion in inverse powers of $m$.

As explained in [3], the low momentum limit

$$\Psi = \exp\left(-\frac{1}{2g_{YM}^2 m}\int \mathrm{Tr}\ B^2\right) \tag{15}$$

provides a probability measure $\Psi^*\Psi$ equivalent to the partition function of the Euclidean two-dimensional Yang-Mills theory with an effective Yang-Mills coupling $g_{2D}^2 \equiv m g_{YM}^2$. Using the results from [6], Karabali, Kim and Nair deduced the area law for the expectation value of the Wilson loop operator

$$\langle\Phi\rangle \sim \exp(-\sigma A) \tag{16}$$

with the string tension following from the results of [6]

$$\sigma = g_{YM}^4 \frac{N^2-1}{8\pi} \tag{17}$$

This formula agrees beautifully with extensive lattice simulations [4], and is certainly consistent with the appearance of a mass gap. Notice that this result is once again in full agreement with the large $N$ 't Hooft expansion.

[2] This comes from a piece of $S_{WZW}$.

# 3 The collective field Hamiltonian

We are interested in studying the planar limit of the Karabali-Nair approach to $YM_{2+1}$. The large $N$ limit is expected to be controlled by a constant $\infty \times \infty$ matrix configuration called the "master field" [7]. Such a configuration should capture correctly both the short and long distance properties, that is, both asymptotic freedom as well as confinement. As already mentioned above, knowing the master field configuration is, in a very precise sense, equivalent to knowing the correct vacuum at large $N$, which is the most remarkable result of the Karabali-Nair approach as seen from the preceding section. Small perturbations around this configuration should lead to the spectrum of glueballs. In the planar limit this spectrum is expected to be equidistant, consisting of an infinite number of non-interacting massive colorless particles.

Can we compute the planar spectrum of $YM_{2+1}$ using the Karabali-Nair scheme? The claim of this letter is that in the low momentum, or equivalently large 't Hooft coupling limit, the planar spectrum of $YM_{2+1}$ is explicitly computable. Because the knowledge of the master field is equivalent to the knowledge of the true vacuum in the large N limit, the crucial property to be used is of the holomorphic invariance of the gauge invariant variables $H \to VH\bar{V}$. This fact, in combination with the known properties of the spectrum of singlet states in the one matrix model is what makes the computation of the planar spectrum possible.

The crucial observation we make here is that the above Karabli-Nair vacuum wave functional consistent with the area law (and as we will see, with the existence of a mass gap), is captured by an effective quadratic matrix model, in the large $N$, large 't Hooft coupling limit. The usual power counting arguments (based on the power expansion in terms of the large $N$ 't Hooft coupling) can be applied to the Karabali-Nair vacuum wave functional by assuming a WKB ansatz, $\Psi = \exp(\Gamma)$, which is consistent with factorization at large N. By assuming a local expansion of $\Gamma$ in terms of gauge invariant observables, such as $J$ currents, (involving possible non-local, $J$-independent kernels), the leading term in the large $N$ 't Hooft coupling is given by a quadratic expression in terms of currents! We will show that this Karabali-Nair vacuum wave functional is reproduced by an effective quadratic matrix model involving renormalized couplings (such as the mass $m$).

In order to get to this result our first aim is to better understand the structure of the Karabali-Nair collective field Hamiltonian. That is the subject of the present section. In the following section we will study the collective field Hamiltonian in the planar limit.

The Karabali-Nair collective field Hamiltonian can be thought of as a string field theory Hamiltonian for a pure QCD string in $2+1$ dimensions. Indeed, the Karabali-Nair variables represent local gauge invariant variables and act operatorially on a true, non-perturbative Fock space of $2+1$ dimensional Yang-Mills theory. Thus this description is intrinsically second-quantized and gauge invariant, and as such does qualify as a QCD string field theory. Note that this second quantized theory does act as an interacting theory in the usual space, in other words this QCD string field theory is not formulated on a loop space, precisely because the string field in this approach (identified with the Karabali-Nair variables) is local. As we will see in the next section, the Hamiltonian of this effective QCD string field theory is generically non-local. A first quantized worldsheet theory remains for the moment elusive; presumably such a theory is interacting, and moreover, can be deduced from the target space second quantized theory studied in this letter. Nevertheless, a purely first quantized description should not be considered satisfactory as it would yield just the spectrum and compute S-matrix-type observables. The second quantized formulation in principle contains full knowledge of the non-perturbative Fock space. In this letter we concentrate on the form of the non-perturbative planar vacuum and the spectrum of gauge invariant excitations around it.

## 3.1 Hamiltonian

The classical mechanics of the $H$ variables is somewhat complicated by the constraint $\det H = 1$. The Hamiltonian is more easily found using traceless variables – for example, it is convenient to use the currents

$J = \frac{c_A}{\pi}\partial H H^{-1}$. The Karabali-Nair Hamiltonian[3] is

$$\mathcal{H}_{KN}[J] = m\int J^a \frac{\delta}{\delta J^a} + m\int_{x,y}\Omega^{ab}(x,y)\frac{\delta}{\delta J^a(x)}\frac{\delta}{\delta J^b(y)} + \frac{\pi}{mc_A}\int \bar{\partial}J^a\bar{\partial}J^a \tag{18}$$

where

$$\Omega^{ab}(x,y) = \frac{c_A\delta^{ab}}{\pi^2(x-y)^2} - i\frac{f_{abc}J^c(y)}{\pi(x-y)} \tag{19}$$

This is derived within a consistent gauge-invariant regularization scheme. The last term in (18) is the potential term, and follows from the precise relation $\bar{\partial}J = \frac{c_A}{2\pi i}M^\dagger B M^{-\dagger}$.

Alternatively, suppose we consider expanding around the constant solution as

$$H = e^{\varphi}, \tag{20}$$

where $\varphi = \varphi^a t^a$ is Hermitian and traceless. The perturbative vacuum is described by $H = 1$, and we are to expand the theory in powers of $\varphi$. Given the parameterization of $H$, one finds

$$H^{-1}\partial H = \partial\varphi + \frac{1}{2}\left[\partial\varphi,\varphi\right] + \frac{1}{6}\left[\left[\partial\varphi,\varphi\right]\varphi\right] + \dots \tag{21}$$

Using the (adjoint) notation[3] $\varphi^{ab} \equiv \varphi^c f^{abc}$ (or equivalently, $f^{abc}\varphi^{bc} = c_A\varphi^a$), we find

$$H^{-1}\partial H = t^a\partial\varphi^b e_{ba}[\varphi] \tag{22}$$

where[4] $e_{ba}[\varphi] = \delta_{ba} - \frac{i}{2}\varphi_{ba} - \frac{1}{6}(\varphi^2)_{ba}\dots$ is a functional of $\varphi$. Note that in this notation, we have a generalized non-Abelian bosonization formula

$$J^a[\varphi] = \frac{ic_A}{\pi}e_{ab}[\varphi]\partial\varphi^b. \tag{23}$$

Moreover one can also establish the following useful formula (see the Appendix):

$$\frac{\delta}{\delta J^a(x)} = \frac{i\pi}{c_A}\int_y D'^{-1}_{ac\ (x,y)}\frac{\delta}{\delta\varphi^c(y)} \tag{24}$$

where $D'_{ab} \simeq \delta_{ab}\partial - \frac{i}{2}f_{abc}\varphi^c\partial + if_{abc}\partial\varphi^c + \dots$.

Given this dictionary between $J$ and $\varphi$ variables, the collective field theory Hamiltonian may also be expressed in terms of the $\varphi$ variables. It has the form

$$\mathcal{H}_{KN}[\varphi] \quad = \quad \int_x P^a[\varphi](x)\frac{\delta}{\delta\varphi^a(x)} + \int_{x,y}Q^{ab}[\varphi](x,y)\frac{\delta}{\delta\varphi^a(x)}\frac{\delta}{\delta\varphi^b(y)} + \frac{\pi}{mc_A}\int_x\bar{\partial}J^a[\varphi]\bar{\partial}J^a[\varphi] \tag{25}$$

The formulae for the functionals $P^a[\varphi]$ and $Q^{ab}[\varphi]$ can be found in the Appendix. Formally, we find point-split versions of:

$$P^a[\varphi](x) \quad = \quad -\frac{g_{YM}^2}{2}\int_z (e^{-1}(x))_{ce}\bar{G}_{(x,z)}H(z)_{ed}G_{(z,x)}(e^{-1}(x))_{da,c} \tag{26}$$

$$Q^{ab}[\varphi](x,y) \quad = \quad -\frac{g_{YM}^2}{2}\int_z (e^{-1}(y))_{be}\bar{G}_{(y,z)}H(z)_{ed}G_{(z,x)}(e^{-1}(x))_{da} \tag{27}$$

As in the Appendix, $H_{ab}$ is the adjoint representation $(e^{-i\varphi})_{ab}$. As we will see, the change of variables from $J$ to $\varphi$ does introduce extra technical problems in the regularizaton of various expressions, but these issues do not effect the final physical result. In particular, the mass gap is not an artefact of regularization, precisely because the collective Hamiltonian was derived using a consistent gauge-invariant regularization scheme.

[3] We distinguish the two representations by explicit indices in the following formulae when necessary. Note that here $\varphi^{ab}$ is real skew-symmetric.

[4] There is an explicit resummation $e_{[\varphi]} = i\varphi^{-1}(e^{-i\varphi} - 1)$, in the adjoint notation.

### 3.2 WZW

Next, let us look at the WZW action. We have

$$S_{WZW} = -\frac{1}{2\pi}\int_M Tr\ H^{-1}\partial H.H^{-1}\bar{\partial}H + \frac{i}{12\pi}\int_P \epsilon^{\mu\nu\lambda}Tr\ H^{-1}\partial_\mu H.H^{-1}\partial_\nu H.H^{-1}\partial_\lambda H \tag{28}$$

where $\partial P = M$ and $H$ is Hermitian. Given the parameterization of $H$, we can then evaluate

$$\begin{aligned} Tr\ H^{-1}\partial H.H^{-1}\bar{\partial}H &= \frac{1}{2}\partial\varphi^a\bar{\partial}\varphi^b e_{ac}[\varphi]e_{bc}[\varphi] & (29)\\ &= \frac{1}{2}\partial\varphi^a\bar{\partial}\varphi^b g_{ba}[\varphi] & (30)\end{aligned}$$

where $g[\varphi] = e_{[\varphi]}e_{[-\varphi]} = I - \frac{1}{12}\varphi^2 + \dots$

The WZW term may be written

$$-\frac{i}{4\pi}\int d^2z ds\ h_{abc}[\varphi]\partial\varphi^a\bar{\partial}\varphi^b\partial_s\varphi^c \tag{31}$$

where $h_{abc} = f_{def}e_{ad}e_{be}e_{cf}$. This term is of course a total derivative, and gives

$$\frac{i}{4\pi}\int d^2z\ \partial\varphi^a\bar{\partial}\varphi^b b_{ba}[\varphi] \tag{32}$$

where $h_{abc} = b_{ab,c} - b_{cb,a} - b_{ac,b}$. ($b = \frac{1}{3}\varphi + \dots$ is antisymmetric).

Thus we arrive at the familiar result that the full WZW action can be written as a sigma model action

$$S_{WZW} = -\frac{1}{4\pi}\int \partial\varphi^a\bar{\partial}\varphi^b G^{WZW}_{ab}(\varphi) \tag{33}$$

where

$$G^{WZW}_{ba}(\varphi) = (g+ib)_{ba}(\varphi) \tag{34}$$

For small $\varphi$ fields, which correspond to $H$ of order one, one can use the standard background field method. We obtain

$$G^{WZW}_{ba}(\varphi) = \left(1 + \frac{i}{3}\varphi - \frac{1}{12}\varphi^2 + \dots\right)_{ba} \tag{35}$$

The calculations presented below can be obtained by following this formalism.

Now, we may proceed to do the redefinition of the wavefunctionals. The effect of this redefinition is to give a new collective Hamiltonian, which is a similarity transform of $\mathcal{H}_{KN}$

$$\mathcal{H}' = e^{c_A S_{WZW}}\mathcal{H}e^{-c_A S_{WZW}} \equiv \mathcal{H}_2 + \mathcal{H}_1 + \mathcal{H}_0 \tag{36}$$

where

$$\mathcal{H}_2 = \int_{x,y} Q^{ab}[\varphi](x,y)\frac{\delta}{\delta\varphi^a(x)}\frac{\delta}{\delta\varphi^b(y)} \tag{37}$$

and

$$\mathcal{H}_1 = \int_x \left[P^a[\varphi](x) - 2c_A\int_y Q^{ab}[\varphi](x,y)\frac{\delta S_{WZW}}{\delta\varphi^b(y)}\right]\frac{\delta}{\delta\varphi^a(x)} \tag{38}$$

and, finally

$$\mathcal{H}_0 = -\frac{c_A}{m\pi}\int_x \bar{\partial}(e_{ab}\partial\varphi^b)\bar{\partial}(e_{ac}\partial\varphi^c) - c_A\int_x P^a[\varphi](x)\frac{\delta S_{WZW}}{\delta\varphi^a(x)} \tag{39}$$

$$-c_A\int_{x,y} Q^{ab}[\varphi](x,y)\left[\frac{\delta^2 S_{WZW}}{\delta\varphi^a(x)\delta\varphi^b(y)} - c_A\frac{\delta S_{WZW}}{\delta\varphi^a(x)}\frac{\delta S_{WZW}}{\delta\varphi^b(y)}\right] \tag{40}$$

These are formal manipulations and one needs to take care to regulate the expressions in a gauge and holomorphic invariant manner. In fact, in general this expression will be *non-local*; however, as we will see later, the real utility of the $\varphi$ variables is that the Jacobian discussed above essentially generates a mass gap. The mass gap is set by the 't Hooft coupling itself, and so there should be a self-consistent low-momentum expansion, in powers of $k/m$.

Let us work in an expansion in powers of $\varphi$. Recall that $e_{ba} = \delta_{ba} - \frac{i}{2}\varphi_{ba} - \frac{1}{6}(\varphi^2)_{ba} \ldots$ and $G_{ab} = (g+ib)_{ab} = \delta_{ab} + \frac{i}{3}\varphi_{ab} + \ldots$. Note that we have

$$\begin{aligned} \frac{\delta S_{WZW}}{\delta\varphi^a(x)} &= \frac{1}{2\pi}\left[\bar{\partial}\partial\varphi^b g_{ab}[\varphi] + \frac{1}{2}\partial\varphi^c\bar{\partial}\varphi^d(G_{ad,c}[\varphi] + G_{ca,d}[\varphi] - G_{cd,a}[\varphi])\right] \\ &= \frac{1}{2\pi}\left[\bar{\partial}\partial\varphi^a + \frac{i}{2}f_{abc}\partial\varphi^c\bar{\partial}\varphi^b + \ldots\right] \end{aligned} \tag{41}$$

Using this expression, we may derive, as an expansion in $\varphi$,

$$\mathcal{H}' = -\frac{g_{YM}^2}{2}\int_x \pi^a(x)\int_y C(x,y)\pi^a(y) + \frac{m_r^2}{2g_{YM}^2}\int\partial\varphi^a\bar{\partial}\varphi^a + \frac{2}{g_{YM}^2}\int\bar{\partial}\varphi^a(-\partial\bar{\partial})\bar{\partial}\varphi^a + \ldots \tag{42}$$

where $C(x,y)$ is in general a non-local kernel whose inverse is, to leading order in $k$, $|k|^2$.

This collective Hamiltonian $\mathcal{H}'$ is also in general non-local. The terms given in eq. (42) represent the part of the Hamiltonian which controls the ground-state in the planar limit; we note however that the parameter $m$ should be understood as a renormalized coupling $m_r$. The quadratic terms determine a dispersion relation of the form

$$\Delta(k) = |k|^2\left(E^2 - |k|^2 - m_r^2\right). \tag{43}$$

This indicates that the field $\varphi$ is not canonically normalized. However, since $\varphi$ has no physical zero-mode[5], the transformation to canonically normalized excitations is non-singular, albeit non-local. We will explore this further in the next section.

# 4 $N = \infty$ reduction

All of the above analysis is valid at any $N$. Let us now consider the theory in the planar limit. For self-consistency of our presentation we briefly review the Eguchi-Kawai reduction [8], which is strictly valid in the $N = \infty$ limit. We discuss the reduction in the continuum and concentrate on the matrix scalar field theory, obviously relevant for our discussion.

Consider the following general local action for a scalar matrix field $\varphi$

$$S(\varphi) = N\int \text{Tr}\,[\frac{1}{2}(\partial_a\varphi)^2 + V(\varphi(x))] \tag{44}$$

The $N = \infty$ reduction is captured by the following recipe:

1) Replace the position dependent scalar matrix field $\varphi(x)$ by

$$\varphi(x) \to e^{iP_a x^a}\varphi_R e^{-iP_a x^a} \tag{45}$$

and

$$\pi(x) \to e^{iP_a x^a}\pi_R e^{-iP_a x^a} \tag{46}$$

where $P_a$ is a diagonal Hermitian matrix $P^a = diag(p_1^a, p_2^a ... p_N^a)$ and

2) replace the derivative operation $\partial_a\varphi(x)$ by

$$\partial_a\varphi(x) \to i[P_a, \varphi_R] \tag{47}$$

[5] Constant $\varphi$ is equivalent, via a constant holomorphic transformation, to $\varphi = 0$.

and

3) finally, replace the continuum action per unit space-time volume by

$$S_R = Tr[-\frac{1}{2}[P_a, \varphi_R]^2 + V_R(\varphi_R)] \tag{48}$$

Then the correlation functions of the continuum theory (in the $N = \infty$ limit) are given by the correlation functions of the reduced theory, after integrations over the eigenvalues of the momenta $p_i$

$$\langle F[\varphi(x)]\rangle \rightarrow \int \prod_{i=1}^{N} d^D p_i \langle F[e^{iP_a x^a} \varphi_R e^{-iP_a x^a}]_R\rangle \tag{49}$$

Following this prescription we can easily write a reduced form of the local part of the collective field Hamiltonian in terms of $\varphi$ variables. This is sufficient, because this part of the collective field Hamiltonian controls the planar vacuum at large 'tHooft coupling, as we will see in what follows.

As we would like to go to large $N$, we consider the continuum Eguchi-Kawai reduction of the approximate local expression of the Hamiltonian (42). The low momentum limit

$$\frac{p_i^2}{m^2} \ll 1 \tag{50}$$

*defines* the large 't Hooft limit, given the fact that the gauge coupling in $2+1$ dimensions is dimensionful. As we noted above, in the large $N$, large 't Hooft coupling limit, the vacuum wave functional is given by an exponent of a quadratic functional in the current variables. We will show that the large $N$ large 't Hooft coupling limit is self-consistent in the sense that it leads to the correct vacuum implying a gap in the spectrum.

To proceed, we introduce the following change of variables in momentum space (a non-local change of variables in real space) that we alluded to in the previous section, in order to get a canonically normalized[6] kinetic term

$$\phi^a(\vec{k}) = \sqrt{\frac{k\bar{k}}{g_{YM}^2}}\ \varphi^a(\vec{k}). \tag{51}$$

We see that in the low momentum limit of the reduced Hamiltonian we simply get

$$\frac{1}{2}\int Tr\left(\pi^2 + m_r^2\phi^2 - \phi[P_a,[P^a,\phi]] + ...\right) \tag{52}$$

where the canonical momentum $\pi = -i\frac{\delta}{\delta\phi}$.

As it stands this Hamiltonian apparently describes $N^2-1$ massive degrees of freedom with a mass proportional to the square of the gauge coupling, as noted originally in [3]. Obviously this does not seem to give the confining spectrum we expect of $YM_{2+1}$! As a matter of fact, in the limit of the zero Yang-Mills coupling the familiar perturbative spectrum of gluons is readily recovered, as we discussed in Section 2. This result is natural as we have expanded $H(x) = \exp(\varphi(x))$ around the "perturbative vacuum" $H = 1$ to quadratic order in $\varphi$. The new interesting feature in this discussion is the presence of the gauge invariant mass term, whose origin was the Jacobian of the transformation to these variables.

Nevertheless we claim that one can gain important insight into the nature of the planar vacuum provided we remember that the master field, in terms of gauge invariant $H = \exp(\varphi)$ variables, is supposed to be a *constant* $\infty \times \infty$ matrix that transforms homogeneously under the residual *constant* transformations $H \rightarrow hHh^\dagger$. In particular, *the vacuum is preserved by constant unitary transformations.* Therefore in the planar limit the matrix $\phi$ will also be an $\infty \times \infty$ matrix that transforms as $\phi \rightarrow h\phi h^\dagger$. We must require invariance under constant unitary transformations on the zero-mode of this field.[7]

[6] Note that we normalize to 1 rather than $N$ here for simplicity.

[7] This is a somewhat subtle point because of the momentum-dependent change of variables (51).

Thus, the dynamics of the planar master field of $YM_{2+1}$ which describes the vacuum in the planar, low momentum limit, in the well-defined momentum expansion of the reduced Hamiltonian, given in terms of $\phi$ matrix variables, is determined by the following *one* matrix model Hamiltonian

$$\frac{1}{2}\int Tr\left(\pi^2 + m_r^2\phi^2 + ...\right) \equiv \frac{1}{2}\int Tr\left(-\frac{\delta^2}{\delta\phi^2} + m_r^2\phi^2 + ...\right) \tag{53}$$

*The vacuum in this approximation is captured by the singlets, invariant under the residual* unitary *transformation* $\phi \to h\phi h^\dagger$*.* This is a huge reduction in the number of states in the large $N$ limit, described by the density of $N \to \infty$ eigenvalues of the matrix $\phi$!

Therefore, due to the holomorphic invariance of the vacuum, the spectrum of the planar limit of $YM_{2+1}$ is determined in the planar, low momentum (large 't Hooft coupling) limit, by the spectrum of the singlet states of this effective one matrix model.

Before proceeding further, let us note that in order to compute correlation functions we would need to include momentum dependent terms in the reduced Hamiltonian in terms of $\varphi$ matrix variables. One obvious technical complication one has to deal with in order to get the leading momentum expressions for various correlation functions is that the momentum-dependent terms in the reduced Hamiltonian are not of the usual kind considered in the literature of the one matrix model (i.e. they are not traces or multi-trace terms involving a single matrix). Consequently, the computation of correlation functions will be significantly harder, although we believe the formalism presented here is sufficient for a discussion of the spectrum. In this letter we restrict our attention to the spectrum.

### 4.1 The vacuum wave functional, one more time

Now we demonstrate that the effective quadratic matrix model captures the correct physics of the vacuum. First, we can easily find the correct vacuum wave functional (which leads to the area law and a successful empirical expression for the string tension) using the above matrix field theory approach. Given the quadratic matrix field Hamiltonian (52), we see that the ground state wave functional is a Gaussian

$$\Phi = \exp\left(-\frac{1}{2}\int \phi\sqrt{m_r^2 - \nabla^2}\phi\right) \tag{54}$$

To compare to the previous discussion, we should consider the effect of transforming back to the non-trivial inner product of the original collective field theory. This amounts to a simple shift of the exponent of the wave functional

$$\Psi = \exp\left(-\frac{1}{2}\int \phi(x)\left(-m_r + \sqrt{m_r^2 - \nabla^2}\right)\phi(y)\right) \tag{55}$$

Conversion to the $J$ variables to linear order in $\phi$ fields gives a wave functional which perfectly matches the expression for the wave functional (14), which, as we have seen above, leads to successful predictions of the area law and the string tension.

Thus we explicitly see that in the large $N$, low momentum, that is, large 't Hooft coupling limit, the effective quadratic matrix model leads to the correct physics of the Yang-Mills vacuum, as described by a vacuum wave functional consistent with the area law. This result should not come as too much of a surprise, since it is essentially the same calculation that we considered in Section 2.1. However, it is a useful check that the reduction has not thrown away anything important about the vacuum.

We are now ready to show that the same effective matrix model describes the correct spectrum of excitations about the vacuum of the large $N$ $2+1$-dimensional Yang-Mills theory.

## 5 The planar spectrum of $YM_{2+1}$

In this section we determine the planar spectrum of $YM_{2+1}$ from the spectrum of the singlet states of a one-matrix model. The one matrix model is a well-studied system [9]. It can be understood from many

points of view of the known large $N$ technology [5][9][10]. For example, it is well known that the planar limit of a quadratic matrix model in the singlet sector is completely captured by the semicircular Wigner-Dyson distribution of the density of eigenvalues ($\lambda$) of the matrix $\phi$

$$\rho_{WD}(\lambda) = \frac{1}{\pi}\sqrt{2\alpha - m_r^2\lambda^2} \tag{56}$$

where $\lambda \in [-\Lambda, \Lambda]$, $\Lambda$ being the point where the square root vanishes, that is $\Lambda = \frac{2\alpha}{m_r^2}$. ($\alpha$ is just a Lagrange multiplier associated with the normalization condition satisfied by the eigenvalue density. In the equivalent fermionic formulation, it is the Fermi energy.) Furthermore, the master field of a Gaussian matrix model [10] can be described in terms of noncommutative probability theory [11] (for which the semicircle distributions play the role completely analogous to the Gaussian distributions of the commutative probability theory) as an operator $\phi = a + \frac{\alpha}{2}a^\dagger$ acting on a Fock space build out of $a$ and $a^\dagger$ operators and the vacuum $|0\rangle$, $a|0\rangle = 0$. $a$ and $a^\dagger$ satisfy the Cuntz algebra $aa^\dagger = 1$.

The spectrum on singlet excitations can be easily determined by perturbing around the ground state value, determined by the semicircle law, assuming $e^{i\omega t}$ time dependence and reading off the normal modes from the resulting wave equation. (For example, this has been done in detail in [12] for the case of the one-matrix model.) Because these calculations are crucial for our main claim, we review them from a couple of different points of view, mainly following the work of [12].

## 5.1 Spectrum of singlets from the collective field theory

As stated above, the density of eigenvalues $\rho(\lambda)$ is the correct variable that describes the dynamics of the large $N$ one matrix model. One introduces first $\rho_k \equiv Tr\exp(-ik\phi)$, and then defines the collective field $\rho(\lambda)$ as the Fourier transform of $\rho_k$. Here $\rho(\lambda)$ is positive definite and normalized as $\int \rho(\lambda)d\lambda = N$.

Suppose that the one matrix model is described by the general Hamiltonian $Tr(\frac{1}{2}\pi^2 + V(\phi))$. The collective field Hamiltonian for $\rho(\lambda)$ and its conjugate momentum $\Pi(\lambda)$ reads as follows

$$\frac{1}{2}\int d\lambda[\rho(\lambda)(\partial_\lambda\Pi(\lambda))^2 + \frac{1}{6}\pi^2\rho(\lambda)^3 + V(\lambda)\rho(\lambda) - \alpha\rho(\lambda)] + \alpha N \tag{57}$$

The classical equations of motion for the collective field theory are

$$\dot\rho(\lambda) = -\partial_\lambda[\rho(\lambda)\partial_\lambda\Pi(\lambda)], \quad \dot\Pi(\lambda) = \frac{1}{2}(\partial_\lambda\Pi(\lambda))^2 + \frac{1}{2}\pi^2\rho(\lambda)^2 + V(\lambda) - \alpha \tag{58}$$

The solution of these equations that satisfies the positive definiteness of the eigenvalue density is such that

$$\partial_\lambda\Pi(\lambda) = 0, \qquad \rho(\lambda) = \rho_{WD}(\lambda). \tag{59}$$

The spectrum can be now determined by small perturbations around this classical ground-state solution. In particular, introduce following [12], $\xi(\lambda) = \partial_\lambda\Pi(\lambda)$ and $\rho(\lambda)^2 = \rho_0(\lambda)^2 + \eta(\lambda)$, and expand the Hamilton equations (58) to first order in $\xi$ and $\eta$. Assuming the periodic time dependence $\exp(-i\omega t)$ for both perturbations, we may eliminate $\xi$ to get the following wave equation for $\eta$

$$\omega^2\eta + \frac{\partial^2\eta}{\partial q^2} = 0 \tag{60}$$

where $q(\lambda) = \int_0^\lambda [2(\alpha - V(x))]^{-\frac{1}{2}}dx$.

The boundary conditions are determined from the constraint on the density of eigenvalues $\int \rho(\lambda)d\lambda = N$ and the vanishing of $\rho_0$ for $\lambda = \pm\Lambda$, from which one infers that $\frac{\partial\eta}{\partial q} = 0$ for $q = q(\Lambda)$. This implies that the wave equation (60) describes a finite length string with free ends (for example, $\eta \sim \cos\left(n\pi\frac{q}{q(\Lambda)}\right)$) from which we infer the spectrum

$$\omega = n\omega_c \tag{61}$$

where the fundamental frequency is given by

$$\omega_c = \frac{\pi}{2}\left(\int_0^{\Lambda} d\lambda\ [2(\alpha - V(\lambda))]^{-\frac{1}{2}}\right)^{-1}. \tag{62}$$

In the case of a quadratic potential $V(x) = \frac{1}{2}x^2$, this can be integrated precisely into a gap-like equation

$$\omega_c = \frac{\pi}{2}\left(\int_0^{\Lambda} d\lambda\ [2\alpha - m_r^2\lambda^2]^{-\frac{1}{2}}\right)^{-1} = m_r \tag{63}$$

Of course, the same result follows by directly perturbing the collective field Hamiltonian and determining the normal modes of the quadratic part [12].

As is well know, the above result (61) can be readily understood from a different, and somewhat more intuitive, point of view. As is well known [9], [13] the singlet sector of the one matrix model can be understood in terms of an effective system describing N free spinless fermions in a quadratic potential. The relevant Hamiltonian is

$$\frac{1}{2}p^2 + V(\lambda) \tag{64}$$

The ground state in this picture corresponds to a filled Fermi sea, with the corresponding Fermi energy $\alpha$ so that

$$N = \frac{1}{\pi}\int d\lambda(2(\alpha - V(\lambda)) \tag{65}$$

The density of states in the semiclassical limit is given by

$$\frac{1}{2\pi}\int d\lambda dp\delta(\alpha - (\frac{1}{2}p^2 + V(\lambda))) = \frac{1}{2\pi}\int d\lambda(2(\alpha - V(\lambda))^{-\frac{1}{2}} \tag{66}$$

For a quadratic potential $V(\lambda) = \frac{1}{2}\lambda^2$ the fundamental frequency $\omega_c = m$ is given as an inverse of the density of states at the Fermi level. Of course, this result agrees with the corresponding collective field theory result. Notice, that $\omega_c$ is nothing but the classical frequency of the corresponding classical trajectory describing the motion of free fermions in the effective potential $V(\lambda)$. Finally, the level degeneracy is also easy to discuss in the fermionic picture.

The crucial point here is that the singlet spectrum of a one matrix model has a gap, determined by the mass parameter $m$, is equidistant and consists of free non-interacting excitations. Going back to the planar limit of $YM_{2+1}$, we see that for the large value of the 't Hooft coupling, the master field of the planar $YM_{2+1}$, a constant $\infty \times \infty$ matrix left invariant under the residual holomorphic transformations, is equivalent to the eigenvalue distribution of the *singlet* sector of a quadratic one matrix model. Therefore the spectrum has a gap determined by the mass $m$ and is equidistant. *The mass gap emerges because of the finiteness of the cut of the semi-circle distribution and the nature of the boundary conditions at the end points of the cut.* This offers an interesting view on confinement.

The Karabali-Nair formalism is completely consistent with Lorentz invariance as shown in [3]. The Lorentz invariant form of the planar $YM_{2+1}$ spectrum is then given by the following dispersion relation describing an equidistant spectrum with an explicit mass gap

$$\omega(\vec{k}) = \sqrt{\vec{k}^2 + m_n^2} \tag{67}$$

where $n = 1, 2, ...,$ $m_n = nm_r$, where $m_r$ is the renormalized gauge invariant mass and the bare mass $m^2$ is determined by the planar 't Hooft coupling $g_{YM}^2 N$ via $m = \frac{g_{YM}^2 N}{2\pi}$. First of all, this formula elucidates the meaning of the parameter $m$ in the planar limit of $YM_{2+1}$ in the large 't Hooft coupling limit. Note that as the 't Hooft coupling is taken to zero we recover the dispersion relations for massless particles. This is in accordance with the recovery of a Coulomb potential at short distances, as shown in [3]. Finally, as we

have seen, the matrix model knows about the vacuum wave functional which in the high momentum limit does describe a theory with a Coulombic potential and in the low momentum limit leads to the area law for the expectation value of the Wilson loop and an explicit formula for the string tension which matches the numerical data.

## 5.2 Matrix model reduction and the planar spectrum

Obviously, the structure of higher order momentum dependent terms in the effective gauge invariant Hamiltonian is in general non-local and very involved. The question that we would like to briefly consider is how the higher order terms influence the leading quadratic result for the mass gap and the spectrum. Even though the higher order corrections are explicitly non-local, we will argue that the existence of a mass gap in the large N, large 'tHooft coupling limit, implies a self-consistent expression for the mass term even upon the inclusion of the higher order terms in $\varphi$.

As we have seen, the planar vacuum at leading order in the 'tHooft coupling is governed by the quadratic part of the collective field Hamiltonian, and the higher order contributions to the wave functionals can be re-organized into an expansion in inverse powers of $m$. Motivated by this observation, suppose we cut-off the momentum integrals in all non-local terms in the collective field Hamiltonian by the mass gap $M$ and then expand in momentum and finally perform the large N reduction. This procedure is difficult to implement technically, yet nevertheless, the general form of the effective reduced Hamiltonian involving $\phi$ matrices, would look as follows

$$\frac{1}{2}\int Tr\left(-\frac{\delta^2}{\delta\phi^2}+V(\phi,m_r)+...\right) \tag{68}$$

The coefficients in $V(\phi)$ would only involve powers of the *renormalized* Karabali-Nair mass parameter $m$, denoted by $m_r$.

Now, by insisting on the holomorphic invariance of the vacuum, as in the quadratic case, we see that only the singlet sector of this general one matrix model, corresponds to the vacuum of $YM_{2+1}$. The spectrum of the singlets has been discussed in the previous section. The fundamental frequency, corresponding to the gap of $YM_{2+1}$ is given by

$$\omega_c=\frac{\pi}{2}(\int_0^{\Lambda(m_r)}[2(\alpha-V(\lambda,m_r))]^{-\frac{1}{2}}d\lambda)^{-1} \tag{69}$$

By choosing $\omega_c$ to correspond to the physical mass gap $M$, and by evaluating the renormalized mass at the scale determined by $M$, we get a self-consistent *gap equation*

$$M=\frac{\pi}{2}(\int_0^{\Lambda(m_r(M))}[2(\alpha-V(\lambda,m_r(M)))]^{-\frac{1}{2}}d\lambda)^{-1} \tag{70}$$

Because of the positive definite nature of the potential, a solution to this equation should exist. The lowest numerical value, in the units of the 't Hooft coupling should correspond to the physical gap of $YM_{2+1}$.

One important point here is that the lowest order, quadratic contribution is not misleading when we try to capture the correct long distance and short distance physics in the planar limit. The ultimate reason for this is that the gauge invariant $H$ variables are local, and that the inner product on the space of states is computable in terms of these variables. The lowest, quadratic order for the master field (described by the dynamics of an effective quadratic matrix model) obviously captures an important piece of the correct ground state (i.e. the master field). The effective matrix model leads to the ground state that is also consistent with the form of the vacuum wave functional obtained in [3], predicting the string tension which turns out to be in excellent agreement with the available lattice data [4]. In our case, the numerical value of the mass gap as given by the spectrum of singlets of the effective matrix model has to be fitted to the lattice data.

### 5.3 Towards the QCD String

Finally we add a couple of comments regarding the relevance of our discussion for a possible string theory of the $2+1$ dimensional Yang Mills theory.

Perhaps the two obvious questions regarding our results in view of the usual intuition connected to the QCD string are [8]: 1) how would one demonstrate, within our framework, the expected Hagedorn behavior of the density of states [14] and also 2) how would one establish the existence of Regge trajectories expected from a QCD string?

To approach answering these questions recall that the Karabali-Nair variables capture all the degrees of freedom of the $2+1$ dimensional Yang Mills theory. Also, the Karabali-Nair Hamiltonian is the *exact* collective field Hamiltonian. Thus, in the confined phase of the theory one expects the usual $O(1)$ scaling of the free energy and in the deconfined phase the $O(N^2)$ behavior. Indeed, the effective matrix model we have used to demonstrate the existence of the mass gap, corresponds to the counting of degrees of freedom expected from a confining phase. On the other hand, the original Karabali-Nair collective field Hamiltonian does describe $O(N^2)$ perturbative degrees of freedom. The presence of an explicit mass gap, indicates that the self-consistent effective scalar matrix field theory derived from the Karabali-Nair collective field description is cut off at the scale determined by the gap $M$, which would be in accordance with the expected confinement/deconfinement transition.

In order to really establish the stringy Hagedorn behavior (and also demonstrate the existence of Regge trajectories) in the planar $YM_{2+1}$ one needs to attach the Lorentz indices to the oscillators of the singlet sector of the effective matrix model. The effective matrix model already describes the correct vacuum in the planar limit and the oscillators acting in the corresponding Fock space should obey the algebra of the usual oscillators of the free string theory, because of the large N factorization. This we think is the key to deriving a QCD string field theory for the $2+1$ dimensional Yang Mills theory and the associated stringy (such as Hagedorn and Regge) features. Work on this important issue is in progress.

## 6 Concluding remarks

To conclude, the spectrum of physical excitations in the planar limit of $YM_{2+1}$ can be deduced using the remarkable local holomorphic variables of Karabali and Nair taken in conjuction with some well known results from large $N$ master field technology. The analytic understanding of the spectrum is possible due to a reduction of the $YM_{2+1}$ Hamiltonian for the large 't Hooft coupling (low momentum limit) to the *singlet* sector of an effective one matrix model. The huge reduction of the degrees of freedom to the singlet sector of the one matrix model is a consequence of the holomorphic invariance of the $YM_{2+1}$ vacuum in the Karabali-Nair representation. Note also that the matrix model captures the form of the vacuum wave functional which leads to the area law and a successful empirical expression for the string tension, reinforcing the self-consistency of our approach.

Obviously the approach presented in this letter is just a first step in the direction of unraveling the full planar limit of $2+1$ dimensional Yang-Mills theory. Given our result we believe that we are now well motivated to study the full matrix collective field theory for the gauge invariant holomorphic loop variables. We also believe that some of the old as well as currently pursued ideas pertaining to the subject of the QCD string [15] such as the question of integrability, the analogy between gauge and chiral fields, as well as target space understanding of the gauge theory/gravity duality should be re-examined in the context of $2+1$ dimensional Yang-Mills theory. The gauge invariant collective field Hamiltonian can be understood as a string field theory of $YM_{2+1}$. It would be obviously very interesting to understand the world-sheet structure underlying this target space description in order to get even closer to the (perhaps not so) elusive QCD string. Last, but not least, we believe that a new way is open for a rational analytic approach to the

[8] We thank Ofer Aharony, David Berenstein and Shiraz Minwalla for discussions regarding these issues.

large $N$ $3+1$ dimensional Yang-Mills theory.

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## Appendix

Here we collect some notation and results. First, define $G_{(x,y)}$ and $\bar{G}_{(x,y)}$

$$\partial G_{(x,y)} = \delta^{(2)}(x-y), \qquad \bar{\partial}\bar{G}_{(x,y)} = \delta^{(2)}(x-y) \tag{71}$$

$$G_{(x,y)} = \frac{1}{\pi}\frac{1}{\bar{x}-\bar{y}}, \qquad \bar{G}_{(x,y)} = \frac{1}{\pi}\frac{1}{x-y} \tag{72}$$

Gauge covariant versions are

$$D^{-1}_{(x,y)} = (\partial + A)^{-1}_{(x,y)} = M(x)M^{-1}(y)G_{(x,y)} \tag{73}$$

$$\bar{D}^{-1}_{(x,y)} = (\bar{\partial} + \bar{A})^{-1}_{(x,y)} = M^{-\dagger}(x)M^{\dagger}(y)\bar{G}_{(x,y)} \tag{74}$$

We thus find

$$\delta M(x) = -\int_y D^{-1}_{(x,y)}\delta A(y)M(y) = -M(x)\int_y G_{(x,y)}(M^{-1}\delta A M)(y) \tag{75}$$

$$\delta M^{\dagger}(x) = \int_y M^{\dagger}(y)\delta\bar{A}(y)\bar{D}^{-1}_{(y,x)} = \int_y (M^{\dagger}\delta\bar{A}M^{-\dagger})(y)\bar{G}_{(y,x)}M^{\dagger}(x) \tag{76}$$

$$\delta J = -\frac{c_A}{\pi}(M^{\dagger}\delta A M^{-\dagger})(x) + \frac{c_A}{\pi}\int_y (M^{\dagger}\delta\bar{A}M^{-\dagger})(y)\partial_x\bar{G}_{(y,x)} - \int_y \left[J(x), (M^{\dagger}\delta\bar{A}M^{-\dagger})(y)\right]\bar{G}_{(y,x)} \tag{77}$$

This equation is used to derive $\mathcal{H}_{KN}[J]$ together with the regulated expression Tr $t^a\bar{D}^{-1}(x,x) = \frac{1}{\pi}\mathrm{Tr}\ t^a(A - M^{-\dagger}\partial M^{\dagger})$.

It also follows that

$$\delta H(x) = -H(x)\int_y G_{(x,y)}(M^{-1}\delta A M)(y) + \int_y (M^{\dagger}\delta\bar{A}M^{-\dagger})(y)\bar{G}_{(y,x)}H(x) \tag{78}$$

But we also have

$$\delta H = Ht^a\delta\varphi^b(e_{[\varphi]})_{ba} = t^a H\delta\varphi^b(e_{[\varphi]})_{ab} \tag{79}$$

and so

$$\delta\varphi^a = 2(e^{-1}_{[\varphi]})_{ba}\mathrm{Tr}\ t^b H^{-1}\delta H = 2(e^{-1}_{[\varphi]})_{ab}\mathrm{Tr}\ t^b \delta H H^{-1} \tag{80}$$

Thus we conclude

$$\frac{\delta\varphi^a(x)}{\delta A^b(y)} = 2i(e^{-1}_{[\varphi]})_{ca}(x)G_{(x,y)}\mathrm{Tr}\ t^c M^{-1}(y)t^b M(y) \equiv -iM^{bc}(y)G_{(y,x)}(e^{-1}_{[\varphi]})_{ca}(x) \tag{81}$$

$$\frac{\delta\varphi^a(x)}{\delta \bar{A}^b(y)} = -2i(e^{-1}_{[\varphi]})_{ac}(x)\mathrm{Tr}\ t^c M^{\dagger}(y)t^b M^{-\dagger}(y)\bar{G}_{(y,x)} \equiv i(e^{-1}_{[\varphi]})_{ac}(x)\bar{G}_{(x,y)}{M^{\dagger}}^{cb}(y) \tag{82}$$

$$\frac{\delta^2\varphi^a(x)}{\delta A^b(y)\delta\bar{A}^c(z)} = M(y)^{bd}G_{(y,x)}(e^{-1})_{da,g}(e^{-1})_{ge}\bar{G}_{(x,z)}{M^{\dagger}(z)}^{ec} \tag{83}$$

Introducing a gauge-invariant point-splitting procedure (insertion of a point-splitting Wilson line), we can use these expressions in the evaluation of the Hamiltonian

$$-\frac{g^2_{YM}}{2}\left[\int_{x,z} W^{cb}(z,x)\frac{\delta^2\varphi^a(y)}{\delta A^c(z)\delta\bar{A}^b(x)}\frac{\delta}{\delta\varphi^a(x)} + \int_{x,y,z} W^{dc}(z,x)\frac{\delta\varphi^b(w)}{\delta A^d(z)}\frac{\delta\varphi^a(y)}{\delta\bar{A}^c(x)}\frac{\delta^2}{\delta\varphi^b(w)\delta\varphi^a(y)}\right] \tag{84}$$

$$\simeq \quad -m\left[\int_x \varphi^a(x)\frac{\delta}{\delta\varphi^a(x)} - \int_{x,y} C(x,y)\frac{\delta^2}{\delta\varphi^a(x)\delta\varphi^a(y)} + \ldots\right] \tag{85}$$

As indicated in the text, another way to proceed is to work with $\mathcal{H}_{KN}[J]$ and convert to the $\varphi$ variables. We then need

$$\frac{\delta}{\delta\varphi^c} = \frac{c_A}{i\pi}\left[e_{ac}\partial - (e_{ab,c} - e_{ac,b})\partial\varphi^b\right]\frac{\delta}{\delta J^a} \tag{86}$$

$$= \frac{c_A}{i\pi}D'_{ca}\frac{\delta}{\delta J^a} \tag{87}$$

(note that ${D'}_{ab} \simeq \delta_{ab}\partial - \frac{i}{2}f_{abc}\varphi^c\partial + if_{abc}\partial\varphi^c + \ldots$), so

$$\frac{\delta}{\delta J^a(x)} = \frac{i\pi}{c_A}\int_y {D'}^{-1}_{ac\ (x,y)}\frac{\delta}{\delta\varphi^c(y)} \tag{88}$$

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